

THE POINCARÉ SERIES FOR THE ALGEBRAS OF JOINT INVARIANTS AND COVARIANTS OF n LINEAR FORMS.

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ABSTRACT. Explicit formulas for computation of the Poincaré series for the algebras of joint invariants and covariants of n linear forms are found. Also, for these algebras we calculate the degrees and asymptotic behaviour of the degrees.

Keywords: classical invariant theory; invariants; Poincaré series; combinatorics

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1. Let V_1 be the complex vector space of linear binary forms endowed with the natural action of the special linear group SL_2 . Consider the corresponding action of the group SL_2 on the algebras of polynomial functions $\mathbb{C}[nV_1]$ and $\mathbb{C}[nV_1 \oplus \mathbb{C}^2]$, where $nV_1 := \underbrace{V_1 \oplus V_1 \oplus \cdots \oplus V_1}_{n \text{ times}}$. Denote by $\mathcal{I}_n = \mathbb{C}[nV_1]^{SL_2}$ and by $\mathcal{C}_n = \mathbb{C}[nV_1 \oplus \mathbb{C}^2]^{SL_2}$ the corresponding algebras of invariant polynomial functions. In the language of classical invariant theory the algebras \mathcal{I}_n and \mathcal{C}_n are called the algebra of join invariants and the algebra of join covariants for the n linear binary forms respectively. A generating set of the algebra \mathcal{I}_n was conjectured by Nowicki [1]. It had been proved later by different authors, for instance see [2], [3]. The algebras $\mathcal{C}_n, \mathcal{I}_n$ are affine graded algebras under the usual degree:

$$\mathcal{C}_n = (\mathcal{C}_n)_0 + (\mathcal{C}_n)_1 + \cdots + (\mathcal{C}_n)_j + \cdots, \mathcal{I}_n = (\mathcal{I}_n)_0 + (\mathcal{I}_n)_1 + \cdots + (\mathcal{I}_n)_j + \cdots,$$

where each of subspaces $(\mathcal{C}_n)_j$ and $(\mathcal{I}_n)_j$ is finite-dimensional. The formal power series

$$\mathcal{P}(\mathcal{C}_n, z) = \sum_{j=0}^{\infty} \dim(\mathcal{C}_n)_j z^j, \mathcal{P}(\mathcal{I}_n, z) = \sum_{j=0}^{\infty} \dim(\mathcal{I}_n)_j z^j,$$

are called the Poincaré series of the algebras \mathcal{C}_n and \mathcal{I}_n . In the paper [4] the following expressions for the Poincaré series of those algebras was derived:

$$\begin{aligned} \mathcal{P}(\mathcal{I}_n, z) &= \sum_{k=1}^n \frac{(-1)^{n-k} (n)_{n-k}}{(k-1)!(n-k)!} \frac{d^{k-1}}{dz^{k-1}} \left(\left(\frac{z}{1-z^2} \right)^{2n-k-1} \right), \\ \mathcal{P}(\mathcal{C}_n, z) &= \sum_{k=1}^n \frac{(-1)^{n-k} (n)_{n-k}}{(k-1)!(n-k)!} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{(1+z)z^{2n-k-1}}{(1-z^2)^{2n-k}} \right), \end{aligned}$$

where $(n)_m := n(n+1) \cdots (n+m-1)$, $(n)_0 := 1$ denotes the shifted factorial.

In the present paper those formulas are reduced to the following forms:

$$\mathcal{P}(\mathcal{I}_n, z) = \frac{N_{n-2}(z^2)}{(1-z^2)^{2n-3}} \text{ and } \mathcal{P}(\mathcal{C}_n, z) = \frac{W_{n-1}(z^2) + nzN_{n-1}(z^2)}{(1-z^2)^{2n-1}},$$

where

$$N_n(z) = \sum_{k=1}^n \frac{1}{k} \binom{n-1}{k-1} \binom{n}{k-1} z^{k-1} \text{ and } W_n(z) = \sum_{k=0}^n \binom{n}{k}^2 z^k,$$

denotes the *Narayana polynomials* and the *Narayana polynomials of type B* respectively.

Also, the degrees of algebras \mathcal{I}_n , \mathcal{C}_n and asymptotic behaviors of the degrees are calculated using the explicit expressions for the Poincaré series.

2. Let us prove several auxiliary combinatorial identities.

Lemma 1. *Let m, k, s be non-negative integers. The generalized Le Jen Shoo identity holds:*

$$\sum_{i=0}^{\min\{k,m\}} \binom{m}{i} \binom{m+2s}{i+s} \binom{k-i+2m+2s}{2m+2s} = \binom{m+k+s}{m+s} \binom{m+k+2s}{m+s}.$$

Proof. Taking into account

$$\binom{m}{i} = 0, \text{ for } i > m, \text{ and } \binom{k-i+2m+2s}{k-i} = 0, \text{ for } i > k,$$

we have

$$\begin{aligned} & \sum_{i=0}^{\infty} \binom{m}{i} \binom{m+2s}{i+s} \binom{k-i+2m+2s}{2m+2s} = \\ &= \sum_{i=0}^k \binom{m}{i} \binom{m+2s}{i+s} \binom{k-i+2m+2s}{2m+2s} + \sum_{i=k+1}^{\infty} \binom{m}{i} \binom{m+2s}{i+s} \binom{k-i+2m+2s}{2m+2s} = \\ &= \sum_{i=0}^k \binom{m}{i} \binom{m+2s}{i+s} \binom{k-i+2m+2s}{2m+2s} + \sum_{i=k+1}^{\infty} \binom{m}{i} \binom{m+2s}{i+s} \cdot 0 = \\ &= \sum_{i=0}^k \binom{m}{i} \binom{m+2s}{i+s} \binom{k-i+2m+2s}{2m+2s} = \\ &= \sum_{i=0}^m \binom{m}{i} \binom{m+2s}{i+s} \binom{k-i+2m+2s}{2m+2s} + \sum_{i=m+1}^{\infty} \binom{m}{i} \binom{m+2s}{i+s} \binom{k-i+2m+2s}{2m+2s} = \\ &= \sum_{i=0}^m \binom{m}{i} \binom{m+2s}{i+s} \binom{k-i+2m+2s}{2m+2s} + \sum_{i=m+1}^{\infty} 0 \cdots \binom{m+2s}{i+s} \binom{k-i+2m+2s}{2m+2s} = \\ &= \sum_{i=0}^m \binom{m}{i} \binom{m+2s}{i+s} \binom{k-i+2m+2s}{2m+2s} = \sum_{i=0}^{\min\{k,m\}} \binom{m}{i} \binom{m+2s}{i+s} \binom{k-i+2m+2s}{2m+2s}. \end{aligned}$$

Now the statement follows immediately from following identity, see [5]:

$$\binom{a+c+d+e}{a+c} \binom{b+c+d+e}{c+e} = \sum_i \binom{a+d}{i+d} \binom{b+c}{i+c} \binom{a+b+c+d+e-i}{a+b+c+d},$$

if we set $a = m, b = m + s, c = s, d = 0$ and $e = k$. □

Lemma 2. *Let $k, n > 1$ be non-negative integers; then*

$$\sum_{i=0}^{\min\{k, n-1\}} (-1)^i \binom{n+i-1}{i} \binom{n+k-2}{k-i} \binom{n+2k-i-1}{2k} = \frac{\binom{n+k-1}{k} \binom{n-2+k}{k}}{k+1}.$$

Proof. We have

$$\begin{aligned} & \sum_{i=0}^{\min\{k, n-1\}} (-1)^i \binom{n+i-1}{i} \binom{n+k-2}{k-i} \binom{n+2k-i-1}{2k} = \\ &= \sum_{i=0}^{n-1} (-1)^i \binom{n+i-1}{i} \binom{n+k-2}{k-i} \binom{n+2k-i-1}{2k} = \\ &= \sum_{i=0}^k (-1)^i \binom{n+i-1}{i} \binom{n+k-2}{k-i} \binom{n+2k-i-1}{2k}. \end{aligned}$$

Note that

$$\binom{n+i-1}{i} \binom{n+k-2}{k-i} = \frac{n+i-1}{n-1} \binom{n+k-2}{n+i-2} \binom{n+i-2}{n-2} = \frac{n+i-1}{n-1} \binom{n+k-2}{n-2} \binom{k}{i},$$

and

$$\frac{n-1}{k+1} \binom{n+k-1}{k} = \binom{n+k-1}{k+1}.$$

So we prove that

$$\sum_{i=0}^k (-1)^i (n-1+i) \binom{k}{i} \binom{n+2k-i-1}{2k} = \binom{n+k-1}{k+1}.$$

Let us put $S_1 = \sum_{i=0}^k (-1)^i (n-1+i) \binom{k}{i} \binom{n+2k-i-1}{2k}$. We have:

$$S_1 = (n-1) \sum_{i=0}^k (-1)^i \binom{k}{i} \binom{n+2k-i-1}{2k} + \sum_{i=0}^k (-1)^i i \binom{k}{i} \binom{n+2k-i-1}{2k}.$$

Using the following identity, see [6], p.8

$$\sum_i (-1)^i \binom{n-i}{m-i} \binom{p}{i} = \binom{n-p}{m},$$

we get:

$$\begin{aligned}
S_1 &= (n-1) \binom{n+k-1}{k} - k \sum_{i=1}^k (-1)^{i-1} \binom{k-1}{i-1} \binom{n+2k-(i-1)-2}{2k} = \\
&= (n-1) \binom{n+k-1}{k} - k \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} \binom{n+2k-2-i}{n-2-i} = \\
&= (n-1) \binom{n+k-1}{k} - k \binom{n+k-1}{n-2} = \\
&= (n-1) \left(\frac{(n+k-1)!}{k!(n-1)!} - \frac{(n+k-1)!k}{(k+1)!(n-1)!} \right) = \frac{n-1}{k+1} \frac{(n+k-1)!}{(k+1)!(n-2)!} = \binom{n+k-1}{k+1}.
\end{aligned}$$

This concludes the proof. \square

Substituting $m = n - 3$ and $s = 1$ into Lemma 1, we obtain:

$$\sum_i \binom{n-3}{i} \binom{n-1}{i+1} \binom{2n+k-i-4}{k-i} = \binom{n+k-1}{n-2} \binom{n-2+k}{n-2}.$$

Multiplying both sides by $\frac{1}{n-1}$, ($n > 2$) and using Lemma 2, we get:

$$\begin{aligned}
(1) \quad & \sum_{i=0}^{\min\{k, n-1\}} (-1)^i \binom{n+i-1}{i} \binom{n+k-2}{k-i} \binom{n+2k-i-1}{2k} = \\
& = \sum_{i=0}^{\min\{k, n-3\}} \binom{n-3}{i} \binom{n-2}{i} \binom{2n+k-i-4}{k-i} \frac{1}{i+1}.
\end{aligned}$$

3. We use the derived above combinatorial identities to simplify expressions for the Poincaré series $\mathcal{P}(\mathcal{I}_n, z)$ and $\mathcal{P}(\mathcal{C}_n, z)$ from [4].

Theorem 1. *The following formulas hold:*

$$\begin{aligned}
(i) \quad \mathcal{P}(\mathcal{I}_n, z) &= \frac{\sum_{k=1}^{n-2} \frac{1}{k} \binom{n-3}{k-1} \binom{n-2}{k-1} z^{2k-2}}{(1-z^2)^{2n-3}}, \\
(ii) \quad \mathcal{P}(\mathcal{C}_n, z) &= \frac{\sum_{k=0}^{n-1} \binom{n-1}{k}^2 z^{2k} + \sum_{k=0}^{n-2} \binom{n-2}{k} \binom{n}{k+1} z^{2k+1}}{(1-z^2)^{2n-1}}.
\end{aligned}$$

Proof. (i) Let us expand function

$$\sum_{k=1}^n \frac{(-1)^{n-k} (n)_{n-k}}{(k-1)!(n-k)!} \frac{d^{k-1}}{dz^{k-1}} \left(\left(\frac{z}{1-z^2} \right)^{2n-k-1} \right),$$

into the Taylor series about z . We have

$$\begin{aligned}\mathcal{P}(\mathcal{I}_n, z) &= \sum_{k=1}^n \frac{(-1)^{n-k} (n)_{n-k}}{(k-1)!(n-k)!} \frac{d^{k-1}}{dz^{k-1}} \left(z^{2n-k-1} \sum_{i=0}^{\infty} \binom{2n-k+i-2}{i} z^{2i} \right) = \\ &= \sum_{k=1}^n \frac{(-1)^{n-k} (n)_{n-k}}{(k-1)!(n-k)!} \frac{d^{k-1}}{dz^{k-1}} \left(\sum_{i=0}^{\infty} \binom{2n-k+i-2}{i} z^{2i+2n-k-1} \right) = \\ &= \sum_{k=1}^n \frac{(-1)^{n-k} (n)_{n-k}}{(k-1)!(n-k)!} \sum_{i=0}^{\infty} \binom{2n-k+i-2}{i} \frac{(2i+2n-k-1)!}{(2i+2n-2k)!} z^{2i+2n-2k}.\end{aligned}$$

Substituting $j = n - k$, we have:

$$\begin{aligned}\mathcal{P}(\mathcal{I}_n, z) &= \sum_{j=0}^{n-1} \frac{(-1)^j (n)_j}{(n-j-1)!j!} \sum_{i=0}^{\infty} \binom{n+j+i-2}{i} \frac{(2i+n+j-1)!}{(2i+2j)!} z^{2i+2j} = \\ &= \sum_{j=0}^{n-1} \frac{(-1)^j (n+j-1)!}{(n-j-1)!j!(n-1)!} \sum_{i=0}^{\infty} \binom{n+i+j-2}{i} \frac{(n+2i+2j-j-1)!}{(2i+2j)!} z^{2i+2j} = \\ &= \sum_{j=0}^{n-1} (-1)^j \binom{n+j-1}{j} \sum_{i=0}^{\infty} \binom{n+i+j-2}{i} \binom{2i+n+j-1}{2i+2j} z^{2i+2j} = \\ &= \sum_{k=0}^{\infty} \sum_{i=0}^{\min\{k, n-1\}} (-1)^i \binom{n+i-1}{i} \binom{n+k-2}{k-i} \binom{n+2k-i-1}{2k} z^{2k}.\end{aligned}$$

Using (1), we get:

$$\begin{aligned}\mathcal{P}(\mathcal{I}_n, z) &= \sum_{k=0}^{\infty} \sum_{i=0}^{\min\{k, n-3\}} \binom{n-3}{i} \binom{n-2}{i} \binom{2n+k-i-4}{k-i} \frac{1}{i+1} z^{2k} = \\ &= \sum_{k=0}^{n-3} \binom{n-3}{k} \binom{n-2}{k} \frac{z^{2k}}{k+1} \sum_{i=0}^{\infty} \binom{(2n-3)+i-1}{i} z^{2i}.\end{aligned}$$

Note that

$$\frac{1}{(1-z^2)^{2n-3}} = \sum_{i=0}^{\infty} \binom{2n-4+i}{i} z^{2i}.$$

This completes the proof.

(ii) Denote by

$$A_n(z) = \sum_{k=1}^n \frac{(-1)^{n-k} (n)_{n-k}}{(k-1)!(n-k)!} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{z^{2n-k-1}}{(1-z^2)^{2n-k}} \right),$$

and let $B_n(z) = \mathcal{P}(\mathcal{C}_n, z) - A_n(z)$. Reasoning as in the proof of (i), we have

$$\begin{aligned} A_n(z) &= \sum_{k=0}^{\infty} \sum_{i=0}^{\min\{k, n-1\}} (-1)^i \binom{n+i-1}{i} \binom{n+k-1}{k-i} \binom{n+2k-i-1}{2k} z^{2k} = \\ &= \sum_{k=0}^{\infty} \binom{n+k-1}{k} z^{2k} \sum_{i=0}^{\min\{k, n-1\}} (-1)^i \binom{k}{i} \binom{n+2k-i-1}{2k} = \sum_{k=0}^{\infty} \binom{n+k-1}{k}^2 z^{2k}. \end{aligned}$$

By using the Le Jen Shoo's identity, we get:

$$\begin{aligned} A_n(z) &= \sum_{k=0}^{\infty} \binom{n-1}{i}^2 \sum_{i=0}^{\min\{k, n-1\}} \binom{2n+k-i-2}{k-i} z^{2k} = \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k}^2 z^{2k} \sum_{k=0}^{\infty} \binom{(n-1)+i-1}{i} z^{2i} = \frac{\sum_{k=0}^{n-1} \binom{n-1}{k}^2 z^{2k}}{(1-z^2)^{2n-1}}. \end{aligned}$$

We see that

$$\begin{aligned} B_n(z) &= \sum_{k=1}^n \frac{(-1)^{n-k} (n)_{n-k}}{(k-1)!(n-k)!} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{z^{2n-k}}{(1-z^2)^{2n-k}} \right) = \\ &= \sum_{k=0}^{\infty} \sum_{i=0}^{\min\{k, n-1\}} (-1)^i \binom{n+i-1}{i} \binom{n+k-1}{k-i} \binom{n+2k-i}{2k+1} z^{2k+1} = \\ &= \sum_{k=0}^{\infty} \binom{n+k-1}{k} z^{2k+1} \sum_{i=0}^{\min\{k, n-1\}} (-1)^i \binom{k}{i} \binom{n+2k-i}{n-1-i} = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} \binom{n+k}{n-1} z^{2k+1}. \end{aligned}$$

Using lemma 1 ($m = n - 2, s = 1$), we have:

$$\begin{aligned} B_n(z) &= \sum_{k=0}^{\infty} \sum_{i=0}^{\min\{k, n-2\}} \binom{n-2}{i} \binom{n}{i+1} \binom{k-i+2n-2}{2n-2} = \\ &= \sum_{k=0}^{n-2} \binom{n-2}{k} \binom{n}{k+1} z^{2k+1} \sum_{k=0}^{\infty} \binom{(n-1)+i-1}{i} z^{2i} = \frac{\sum_{k=0}^{n-2} \binom{n-2}{k} \binom{n}{k+1} z^{2k+1}}{(1-z^2)^{2n-1}}. \end{aligned}$$

Thus

$$\mathcal{P}(\mathcal{C}_n, z) = A_n(z) + B_n(z) = \frac{\sum_{k=0}^{n-1} \binom{n-1}{k}^2 z^{2k} + \sum_{k=0}^{n-2} \binom{n-2}{k} \binom{n}{k+1} z^{2k+1}}{(1-z^2)^{2n-1}}.$$

□

Let us rewrite the expressions in terms of the Narayana polynomials $N_n(z)$ and the Narayana polynomials of type B $W_n(z)$ where

$$N_n(z) = \sum_{k=1}^n \frac{1}{k} \binom{n-1}{k-1} \binom{n}{k-1} z^{k-1} \text{ and } W_n(z) = \sum_{k=0}^n \binom{n}{k}^2 z^k.$$

We get

$$\mathcal{P}(\mathcal{I}_n, z) = \frac{N_{n-2}(z^2)}{(1 - z^2)^{2n-3}} \text{ and } \mathcal{P}(\mathcal{C}_n, z) = \frac{W_{n-1}(z^2) + nzN_{n-1}(z^2)}{(1 - z^2)^{2n-1}}.$$

4. The transcendence degrees over \mathbb{C} for the algebras $\mathcal{I}_n, \mathcal{C}_n$ is equal to order of the pole for $\mathcal{P}(\mathcal{I}_n, z), \mathcal{P}(\mathcal{C}_n, z)$ respectively, see [8]. Note that for all n $N_n(1) \neq 0$ and $W_n(1) \neq 0$. These arguments proves

Theorem 2. *The following formulas hold*

$$\begin{aligned} (i) \quad & \text{tr deg}_{\mathbb{C}} \mathcal{I}_n = 2n - 3, \\ (ii) \quad & \text{tr deg}_{\mathbb{C}} \mathcal{C}_n = 2n - 1. \end{aligned}$$

Let $R = R_0 \oplus R_1 \oplus \cdots$ be a finitely generated graded complex algebra, $R_0 = \mathbb{C}$. Denote by

$$\mathcal{P}(R, z) = \sum_{j=0}^{\infty} \dim R_j z^j,$$

its Poincaré series. Letting r be the transcendence degree of the quotient field of R over \mathbb{C} , the number

$$\deg(R) := \lim_{z \rightarrow 1} (1 - z)^r \mathcal{P}(R, z),$$

is called the *degree of the algebra* R . The first two terms of the Laurent series expansion of $\mathcal{P}(R, z)$ at the point $z = 1$ have the following form

$$\mathcal{P}(R, z) = \frac{\deg(R)}{(1 - z)^r} + \frac{\psi(R)}{(1 - z)^{r-1}} + \cdots$$

The numbers $\deg(R), \psi(R)$ are important characteristics of the algebra R . For instance, if R is an algebra of invariants of a finite group G then $\deg(R)^{-1}$ is order of the group G and $2 \frac{\psi(R)}{\deg(R)}$ is the number of pseudo-reflections in G , see [7].

We know explicit forms for the Poincaré series for the algebras of joint invariants and covariants of n linear forms. Thus we can prove the following statement.

Theorem 3. *The degrees of the algebras of joint invariants and covariants of n linear forms are equal to*

$$\begin{aligned} (i) \quad & \deg(\mathcal{P}(\mathcal{I}_n, z)) = \frac{N_{n-2}(1)}{2^{2n-3}} = \frac{\binom{2n-4}{n-2}}{(n-1)2^{2n-3}}, \\ (ii) \quad & \deg(\mathcal{P}(\mathcal{C}_n, z)) = \frac{\binom{2n-2}{n-1}}{2^{2n-2}}, \end{aligned}$$

Proof. (i) Using Theorem 1 and Theorem 2, we have:

$$\begin{aligned} \deg(\mathcal{I}_n) &= \lim_{z=1} (1-z)^{2n-3} \mathcal{P}(\mathcal{I}_n, z) = \lim_{z=1} (1-z)^{2n-3} \frac{\sum_{k=1}^{n-2} \frac{1}{k} \binom{n-3}{k-1} \binom{n-2}{k-1} z^{2k-2}}{(1-z^2)^{2n-3}} = \\ &= \frac{N_{n-2}(1)}{2^{2n-3}} \end{aligned}$$

Note that the number $N_{n-2}(1)$ equal to the Catalan numbers, see [10]. It now follows that

$$\deg(\mathcal{I}_n) = \frac{\binom{2n-4}{n-2}}{(n-1)2^{2n-3}}$$

(ii) We have

$$\begin{aligned} \deg(\mathcal{C}_n) &= \lim_{z=1} (1-z)^{2n-1} \mathcal{P}(\mathcal{C}_n, z) = \\ &= \lim_{z=1} (1-z)^{2n-1} \frac{\sum_{k=0}^{n-1} \binom{n-1}{k}^2 z^{2k} + \sum_{k=0}^{n-2} \binom{n-2}{k} \binom{n}{k+1} z^{2k+1}}{(1-z^2)^{2n-1}} = \\ &= \frac{\binom{2n-2}{n-1} + nN_{n-1}(1)}{2^{2n-1}} = \frac{\binom{2n-2}{n-1}}{2^{2n-2}} \end{aligned}$$

□

Note that asymptotically, the Catalan numbers grow as

$$C_n = \frac{1}{n+1} \binom{2n}{n} \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}.$$

It is easy to calculate asymptotic behaviours of the degrees of the algebras \mathcal{I}_n i \mathcal{C}_n :

Corollary 1. *Asymptotic behaviours of the degrees of the algebras of joint invariants and covariants of n linear forms as $n \rightarrow \infty$ are follows*

$$\deg(\mathcal{I}_n) \sim \frac{1}{2\sqrt{\pi n^3}} \text{ and } \deg(\mathcal{C}_n) \sim \frac{1}{\sqrt{\pi n}}.$$

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